

Symmetry Breaking, Duality and Fine-Tuning in Hierarchical Spin Models *

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We discuss three questions related to the critical behavior of hierarchical spin models: 1) the hyperscaling relations in the broken symmetry phase; 2) the combined use of dual expansions to calculate non-universal quantities; 3) the fine-tuning issue in approximately supersymmetric models.

1. Introduction

The Monte Carlo method has played a major role in the understanding of quantum field theory problems for which there are no known reliable series expansions. However, the errors associated with this method usually decrease like $t^{-1/2}$ where t is the CPU time used for the calculation. This often prohibits obtaining high-accuracy results. If in the next decades a better knowledge of the fundamental laws of physics has to rely more and more on precision tests, one should complement the Monte Carlo methods with new computational tools which emphasize numerical accuracy. This line of reasoning is a motivation to use “hierarchical approximations” (an example is Wilson’s approximate recursion formula [1]) as a starting point since the RG transformation can be handled easily. Before attacking the hard problem of the improvement of such approximations, we would like to show that they allow spectacular numerical accuracy, namely errors decreasing like e^{-At^u} , for some positive constant A of order 1 when t is measured in minutes of CPU time and $0.5 \leq u \leq 1$.

For definiteness, we consider Dyson’s hierarchical model where the total spin in boxes of size 2^l are coupled with a strength $(\frac{c}{4})^l$, with c a free parameter set to $2^{(D-2)/D}$ in order to approximate a nearest neighbor scalar model in D -dimensions

(see Ref. [2] for details). These interactions are obviously not ultralocal. Models with Fermi fields [3] can be constructed similarly by replacing $D-2$ by $D-1$ in c . In addition one needs to specify a local measure $W_0(\phi)$, for instance of the Landau-Ginzburg type ($W_0(\phi) = e^{-A\phi^2 - B\phi^4}$) or of the Ising type. Under a block spin transformation, the local measure changes according to

$$W_{n+1}(\phi) \propto e^{\frac{\beta}{2}(\frac{c}{4})^{n+1}\phi^2} \times \int d\phi' W_n(\frac{\phi - \phi'}{2}) W_n(\frac{\phi + \phi'}{2}). \quad (1)$$

This recursion formula can be reexpressed in Fourier representation as

$$R_{n+1}(k) \propto \exp(-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2})(R_n(\frac{\sqrt{c}k}{2}))^2, \quad (2)$$

where $R_n(k)$ is the Fourier transform of $W_n(\phi)$ with an appropriate rescaling. It was found [2] that polynomial approximations of reasonably small degree provide very accurate results in the symmetric phase. The initial coefficients are the only integrals which needs to be calculated, after we only perform *algebraic* manipulations. The effects of finite dimensional truncations decay faster than $e^{-A_1\sqrt{t}}$ and the finite-size effects like e^{-A_2t} . Direct fits of the susceptibility give a value [4] of the critical exponent $\gamma = 1.299140730159$. All the digits of this result are confirmed by a calculation of the eigenvalues of the linearized RG transformation about the non-trivial fixed point [5] of Eq. (2).

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Unfortunately, is not possible to get rid of the finite size effects in the broken symmetry phase [4] by iterating Eq. (2) enough times because as one moves away from the fixed point, rapid oscillations appear. Namely, $R_n(k) \simeq \cos(mc^{n/2}k)$, where m is approximately the magnetization at zero external field. It is easy to see from Eq. (2), that for n large enough, the polynomial approximation breaks down as the argument of the exponential becomes too large. It is nevertheless possible to take advantage of the short number of iterations for which the low temperature scaling is observed to obtain reliable extrapolations, first to infinite volume and non-zero external field and then to zero external field. Proceeding this way, one can calculate the connected q -point functions at zero momentum $G_q^c(0)$, for $q = 1, 2$ and 3 and for various values of β . The scaling and hyper-scaling relations together with $\eta = 0$ imply

$$G_q^c(0) \propto (\beta - \beta_c)^{(\gamma/2)(D(1-q/2)-q)}, \quad (3)$$

where γ is the value calculated in the symmetric phase. Our numerical estimate of the exponents [6] agree with the predicted values of the exponents with an accuracy of 10^{-3} , which is respectable when compared to results obtained with conventional methods, but large compared to the 10^{-12} errors obtained in the symmetric phase. In order to improve the accuracy of these results, one could increase the degree of the polynomial used for the approximation, however a short calculation shows that with this procedure, the errors only decrease like $1/t$. A more appealing possibility consists in factorizing $R_n(k)$ into two part: one rapidly oscillating and which requires an exact treatment and another slowly varying which can be approximated by polynomials. This second possibility is presently under investigation.

We now discuss the possibility of replacing the numerical evaluation (which has to be repeated for each choice of the input parameters) by a suitable series expansion (which could be done once for all). Our goal is to obtain an analytical formula for the non-universal quantities ($A_0, A_1 \dots$) appearing in the parametrization of the susceptibility:

$$\chi \simeq (\beta_c - \beta)^{-\gamma} (A_0 + A_1(\beta_c - \beta)^\Delta + \dots). \quad (4)$$

The linearization method used to calculate the universal exponents γ and Δ does not provide a way to calculate the non-universal quantities ($A_0, A_1 \dots$). Indeed, their values “build up” during the crossover between the unstable IR fixed point and the high-temperature (HT) fixed point. A proper description of this crossover requires non-linear expansions about *both* fixed points. This problem can be solved completely [7] for the simplified recursion relation for the magnetic susceptibility

$$\chi_{n+1} = \chi_n + (\beta/4)(c/2)^{n+1} \chi_n^2, \quad (5)$$

which was used in Ref. [2] to estimate the finite volume effects. In this simplified model, one can construct non-linear functions y and \tilde{y} which transform covariantly (multiplicatively) under a RG transformation. These two functions are expressed as an expansion about the IR and HT fixed point respectively. It turns out that there exists [7] a duality relation between the series expansions of the two functions. It would be very interesting to see if similar methods could be applied for non-linear sigma models or gauge theories. The quantity A_0 is a function of $(\beta_c - \beta)$ with a discrete scale invariance and it can be expanded in Fourier modes as in the original model [8]. For practical purposes, the contribution of the non-zero modes is exponentially suppressed and in very good approximation:

$$A_0 = \frac{1}{\ln \lambda} \int_{z_a}^{\lambda z_a} (dz/z) z^\gamma \tilde{y}[1 - y^{-1}(z)], \quad (6)$$

where λ is the eigenvalue associated with the unstable direction of the IR fixed point. The lower value z_a of the integration interval is arbitrary and we can choose it at our convenience. We have compared the approximate values $A_0(m, \tilde{m})$ obtained from expansions with m terms for y^{-1} and \tilde{m} terms for \tilde{y} with an accurate value of A_0 and found that the errors go approximately like $\exp(-K_1(m + \tilde{m}) + K_2(m - \tilde{m})^2)$. This implies that for $m + \tilde{m}$ fixed, it is very advantageous to pick the “self-dual” option $m \simeq \tilde{m}$.

We now discuss the fine-tuning question. The fact that the bare mass of a scalar field theory requires a fine-tuning in order to keep the

renormalized mass small in cut-off units is usually regarded as an argument against fundamental scalars. A possible resolution of this feature consists in adding degrees of freedom in such a way that the quantum fluctuations cancel, making small scalar masses a more natural outcome. In a recent preprint [3], we presented two models with an approximate supersymmetry and which can be solved non-perturbatively with the numerical methods discussed above. The bosonic part of these models is Dyson's hierarchical model. The free action for N massless scalar fields $\phi_x^{(i)}$ reads

$$S_B^{free} = \frac{1}{2} \sum_{x,y,i} \phi_x^{(i)} D_{xy}^2 \phi_y^{(i)}, \quad (7)$$

where x and y run over the sites and i from 1 to N . The action for free massless fermions reads

$$S_F^{free} = \sum_{x,y,i} \bar{\psi}_x^{(i)} D_{xy} \psi_y^{(i)}, \quad (8)$$

where the $\psi_x^{(i)}$ and $\bar{\psi}_x^{(i)}$ are Grassmann numbers. The explicit form of D_{xy} , its square D_{xy}^2 and the supersymmetry of the free action are given in [3].

The Grassmann nature of the fermionic fields restricts severely the type of interactions allowed. For instance, for one flavor the most general local measure (without any supersymmetry considerations) is

$$\mathcal{W}(\phi, \psi, \bar{\psi}) = W(\phi) + \psi \bar{\psi} A(\phi) \quad (9)$$

A more interesting local measure with two-flavors and a R-symmetry is given in [3]. The RG transformation can be expressed as a finite number of convolutions which can be calculated accurately in Fourier transform with polynomial approximations as in the bosonic case (see [3] for explicit formulas). Some numerical results are summarized in Fig. 1 where the renormalized mass is plotted as function of the bare mass (both in cut-off units). One sees that in the purely bosonic interacting model (LG), the quantum corrections to m_R^2 are positive while for a Gaussian model coupled to fermions with a Yukawa coupling, the corrections are negative. For visual reference, the bosonic Gaussian result, where the two quantities are obviously equal, is also plotted (straight

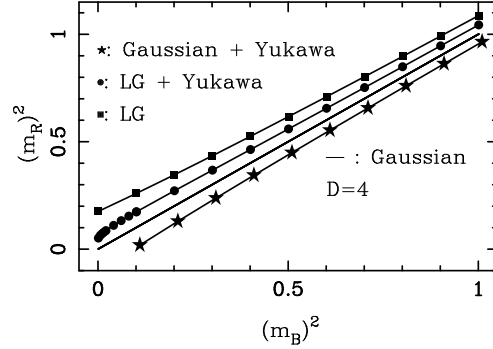


Figure 1. The renormalized mass (m_R) as a function of the bare mass (m_B) for various models.

line). We would like to know if by combining the two interactions in an appropriate way, it possible to cancel the two corrections. At the one-loop level, a simple relation ($g_y^2 = 8\lambda_4$) guarantees that m_R goes to zero when m_B^2 goes to zero. However, we found numerically that for such a choice, $m_R^2 \simeq 0.044$ (in cutoff units) when m_B^2 goes to zero. It is possible to fine-tune g_y in order to get $m_R = 0$ and the exact critical value of g_y is about 50 percent larger than the perturbative one. We are presently trying to build models with better naturalness properties.

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